4.6 – Rational Functions

Def. Let $f(x) = \frac{p(x)}{q(x)}$, where p & q are polynomials and $q(x) \neq 0$. We call f a rational function.

Examples. ① $f(x) = \frac{x-1}{x+2}$; $f(x) = \frac{x^2}{x^2-x-2}$ — degree test* ($n = m$)

② $f(x) = \frac{x-1}{x^2-2x-3}$; $f(x) = \frac{x+3}{x^2-9}$ ($n < m$)

③ $f(x) = \frac{x^2-4x+3}{x+1}$; $f(x) = \frac{x^3+2x^2-15x}{x^2-5x-14}$ ($n > m$)

ASYMPTOTE SCENARIOS

	VA	HA	SA
①	possible	$y = \frac{a}{b}$	NO
②	possible	$y = 0$	NO
③	possible	NO	only if $n = m+1$

* $n = \deg(p)$, $m = \deg(q)$

Tests for Asymptotes : $f(x) = \frac{p(x)}{q(x)}$

1. Vertical (VA)	If $q(c) = 0$ and it is not <u>removable</u>,* then f has a VA @ $\boxed{x = c}$.
2. Horizontal (HA)	<u>3 cases</u> (let p & q have degrees n & m and $LC_p = a_n$ & $LC_q = b_m$): (a) $n = m \Rightarrow \boxed{y = \frac{a_n}{b_n}}$ (b) $n < m \Rightarrow \boxed{y = 0}$ (c) $n > m \Rightarrow$ NONE
3. Slant (SA)	If $n = m+1$,* then $\boxed{y = Q(x)}$.

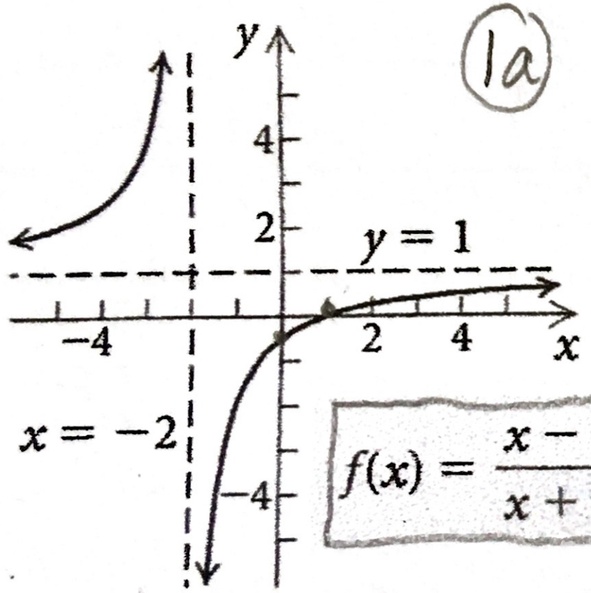
* $x-c$ is factor of $p(x)$ in which case f has a "hole" @ $x = c$.

* If $n \neq m+1$, then $y = Q(x)$ is a nonlinear asymptote (NA).

4.6 - EXAMPLE (1)

● Domain: $(-\infty, -2) \cup (-2, \infty)$;
x-intercept: $(1, 0)$,
y-intercept: $\left(0, -\frac{1}{2}\right)$;

(1a)

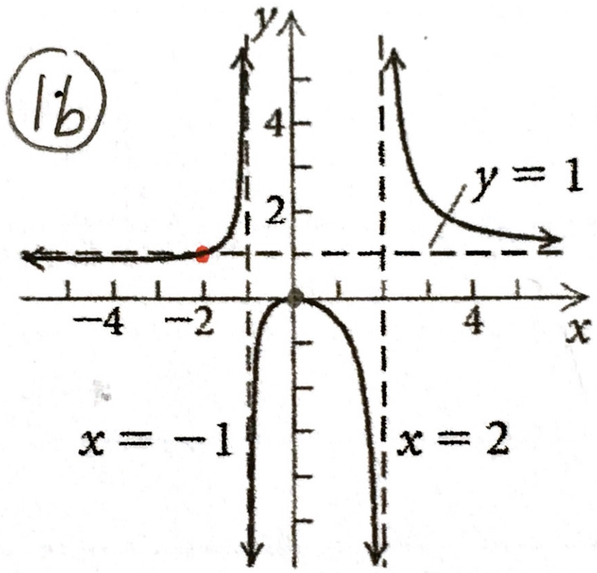


$$f(x) = \frac{x-1}{x+2}$$

(n = m = 1) no holes

● Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$;
x-intercept: $(0, 0)$,
y-intercept: $(0, 0)$;

(1b)



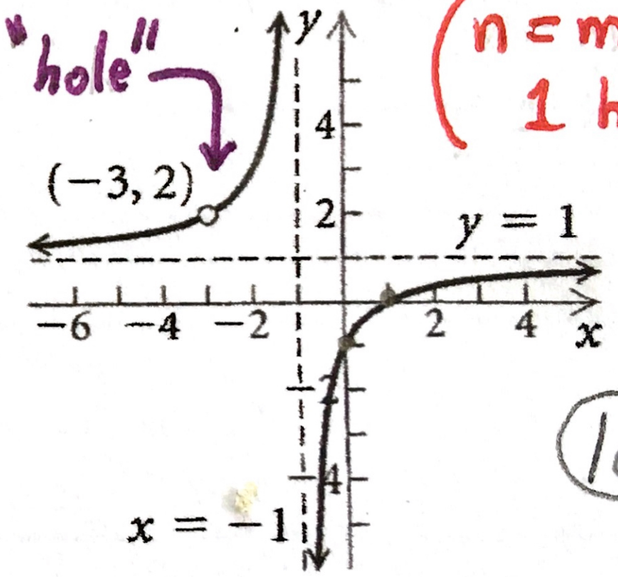
$$f(x) = \frac{x^2}{x^2 - x - 2}$$

(n = m = 2) no holes

● Domain:
$(-\infty, -3) \cup (-3, -1) \cup (-1, \infty)$; x-intercept:
$(1, 0)$, y-intercept: $(0, -1)$;

"hole" (n = m = 2, 1 hole)

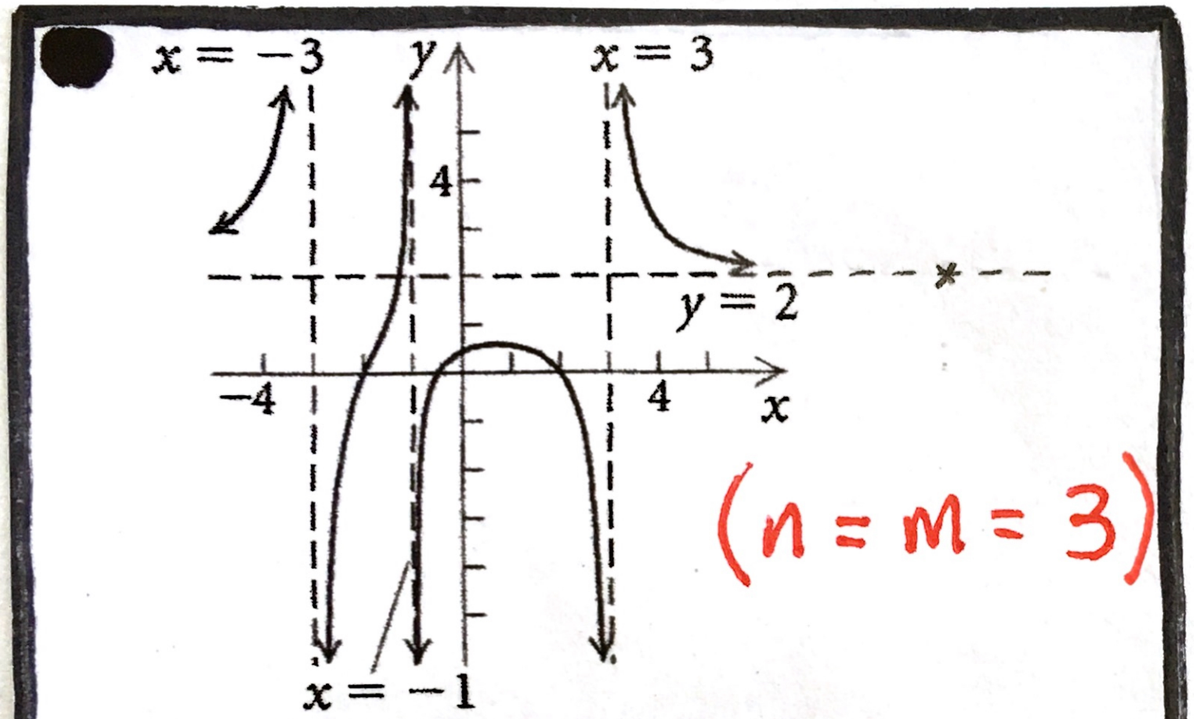
(1c)



$$f(x) = \frac{x^2 + 2x - 3}{x^2 + 4x + 3}$$

$$= \frac{(x+3)(x-1)}{(x+3)(x+1)}$$

$$= \frac{x-1}{x+1}, \quad x \neq -3 \text{ (hole)}$$



(n = m = 3)

$$f(x) = \frac{2x^3 + x^2 - 8x - 4}{x^3 + x^2 - 9x - 9}$$

1d

$$= \frac{(x+2)(2x+1)(x-2)}{(x+3)(x+1)(x-3)}$$

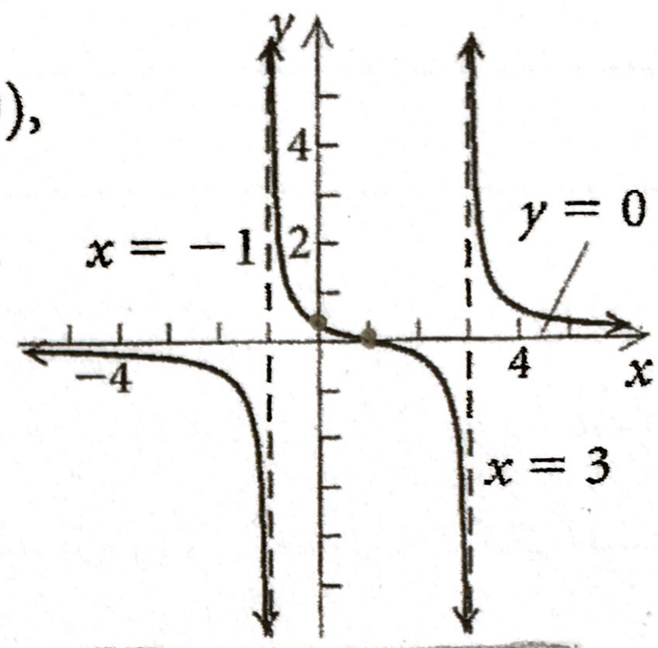
no holes

4.6 – EXAMPLE (2)

(2a)

● Domain: $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$; x-intercept: $(1, 0)$, y-intercept: $\left(0, \frac{1}{3}\right)$;

$n < m$
$n = 1$
$m = 2$



no holes

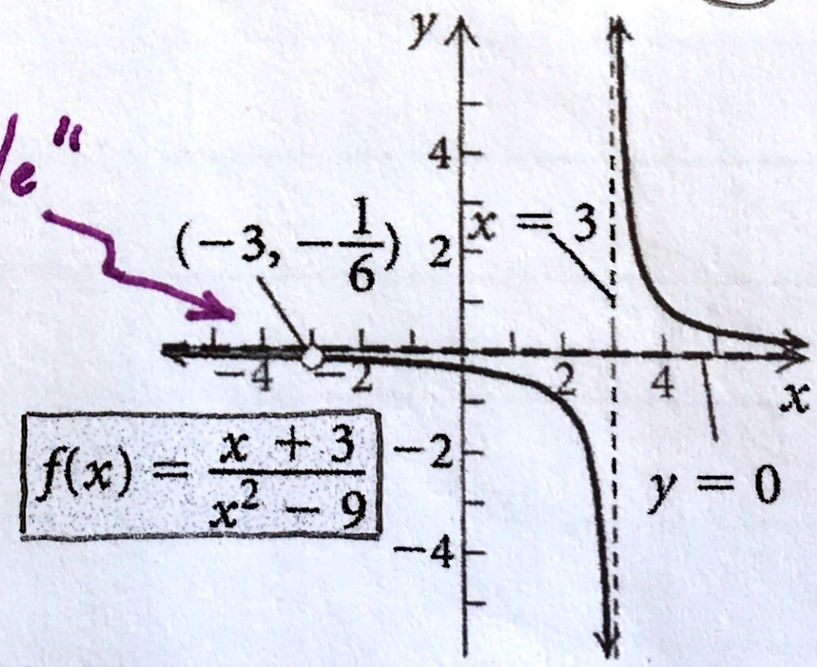
$$f(x) = \frac{x-1}{x^2-2x-3} = \frac{x-1}{(x-3)(x+1)}$$

(2b)

● Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$; no x-intercepts, y-intercept: $\left(0, -\frac{1}{3}\right)$;

"hole"

$n < m$
$n = 1$
$m = 2$
1 hole

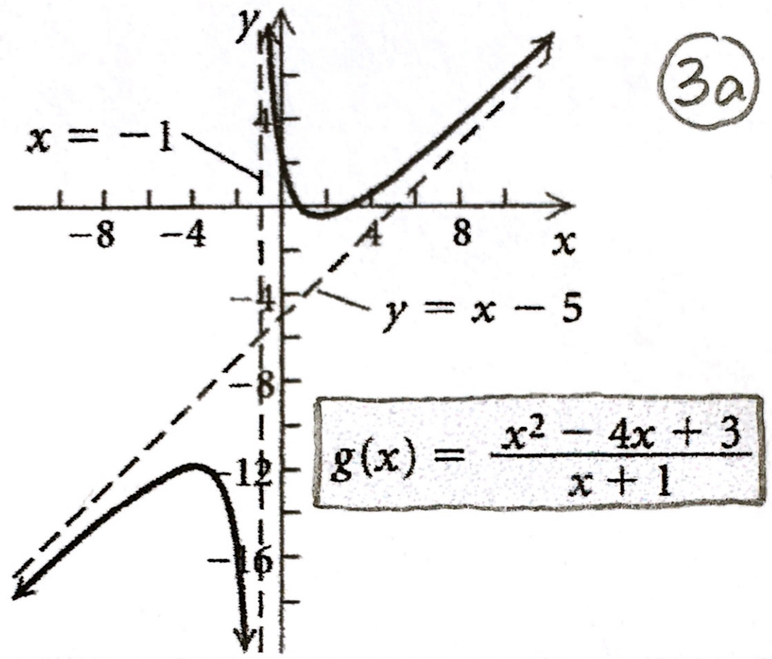


$$f(x) = \frac{x+3}{x^2-9}$$

4.6 – EXAMPLE ③

● Domain: $(-\infty, -1) \cup (-1, \infty)$; x-intercepts: $(1, 0)$ and $(3, 0)$, y-intercept: $(0, 3)$;

$\begin{pmatrix} n > m \\ n = m+1 \\ m = 1 \end{pmatrix} = 2$



3a

$g(x) = \dfrac{x^2 - 4x + 3}{x + 1}$

no holes $= \dfrac{(x-1)(x-3)}{x+1}$

NOTE: The SA is $Q(x)$.

$$\begin{array}{r|rr} -1 & 1 & -4 & 3 \\ & & -1 & 5 \\ \hline & 1 & -5 & 8 \end{array}$$

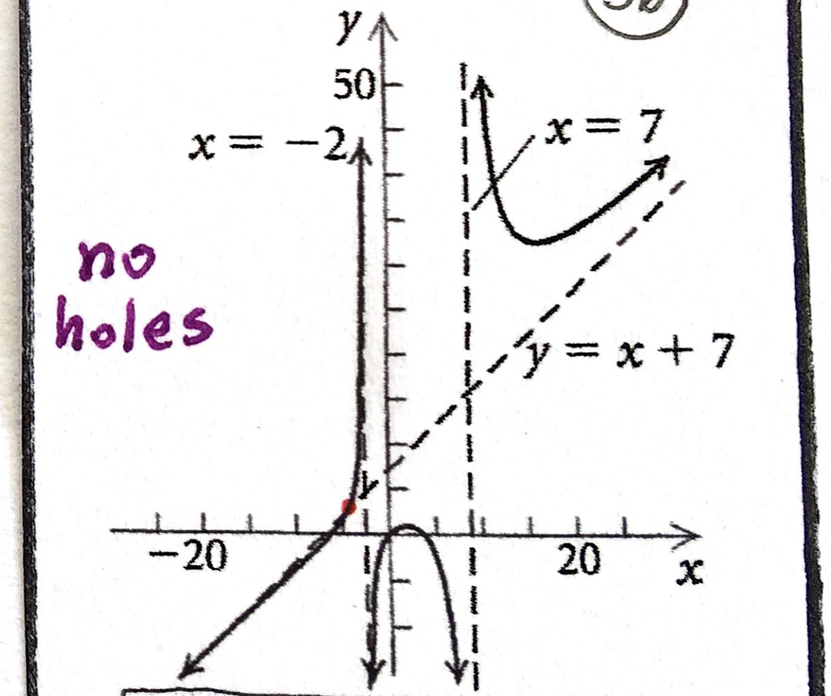
$Q(x)$ ↗ ↖ $R(x)$

Since $Q(x) = x - 5$, SA is $y = x - 5$.

$\begin{pmatrix} n > m \\ n = m+1 \\ m = 2 \end{pmatrix} = 3$

● Domain: $(-\infty, -2) \cup (-2, 7) \cup (7, \infty)$; x-intercepts: $(-5, 0)$, $(0, 0)$, and $(3, 0)$, y-intercept: $(0, 0)$;

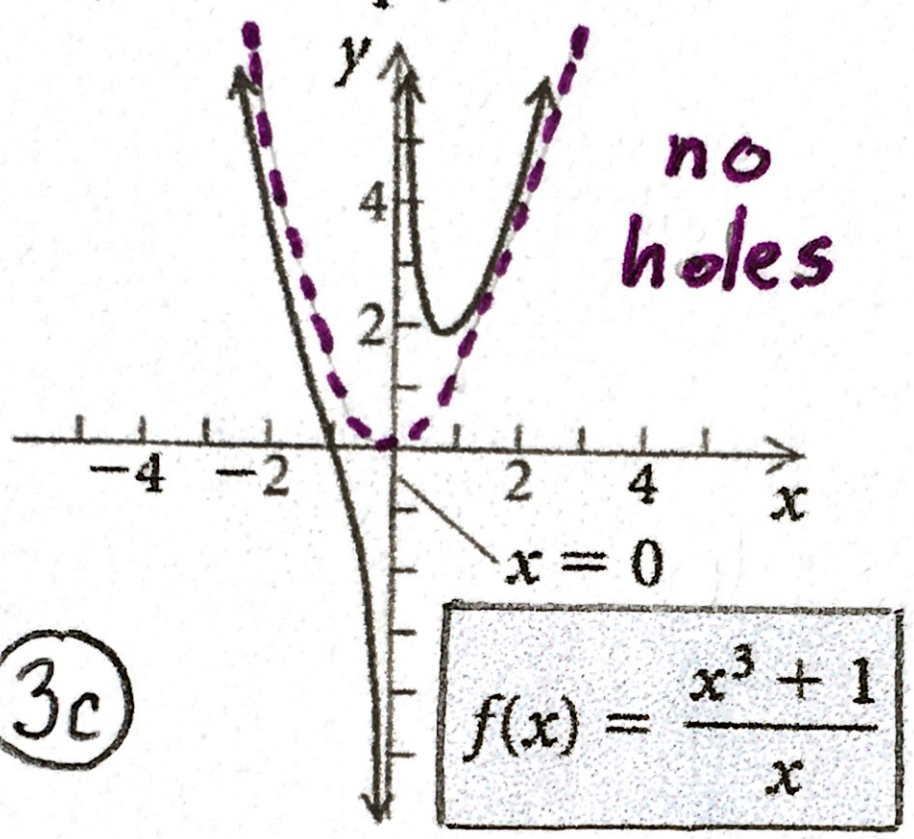
3b



$f(x) = \dfrac{x^3 + 2x^2 - 15x}{x^2 - 5x - 14}$

3

71. Domain:
$(-\infty, 0) \cup (0, \infty)$;
x-intercept: $(-1, 0)$,
no y-intercept;



no holes

3c

$$f(x) = \frac{x^3 + 1}{x}$$

$= x^2 + \frac{1}{x}$

$Q(x)$ ↑ ↑ $\frac{R(x)}{d(x)}$

Since $Q(x) = x^2$, NA is $y = x^2$.

$$\left.\begin{array}{c} n > m \\ n = m + 2 \\ m = 1 \end{array}\right) = 3$$

4.6 (cont'd) – EXAMPLES

I. Find the domain, asymptotes and holes.

(31) $f(x) = \dfrac{4x+1}{2x-6}$

(33) $f(x) = \dfrac{3}{x^2-5}$

(35) $f(x) = \dfrac{x^4}{x^2+3x-10}$

(39) $f(x) = \dfrac{3x(x+2)}{(x+2)(x-1)}$

(41) $f(x) = \dfrac{x^2-9}{x+3}$

(37) $f(x) = \dfrac{x^2+2x+1}{2x^2-3x-5}$

II. Graph. Show all asymptotes and holes.

(97) $f(x) = \dfrac{2x-4}{x-1}$

(79) $f(x) = \dfrac{2x^2+9x+9}{2x^2+7x+6}$

(67) $f(x) = \dfrac{4x+1}{x^2-4}$

⊙ $f(x) = \dfrac{x^2-4}{x-1}$

GRAPHING A RATIONAL FUNCTION

Let $f(x) = \frac{p(x)}{q(x)}$ define a rational function in *lowest* terms. To sketch its graph, follow these steps:

STEP 1: Find all vertical asymptotes.

STEP 2: Find all horizontal or oblique asymptotes.

STEP 3: Find the y-intercept, if possible, by evaluating $f(0)$.

STEP 4: Find the x-intercepts, if any, by solving $f(x) = 0$. (These will correspond to the zeros of the numerator $p(x)$.)

STEP 5: Determine whether the graph will intersect its nonvertical asymptote $y = b$ by solving $f(x) = b$, where b is the y-value of the horizontal asymptote, or by solving $f(x) = mx + b$, where $y = mx + b$ is the equation of the oblique asymptote.

STEP 6: Plot selected points as necessary. Choose an x-value in each interval of the domain determined by the vertical asymptotes and x-intercepts.

STEP 7: Complete the sketch.

STEP 8: Plot any holes.

Graph $f(x) = \dfrac{2x + 1}{x - 3}$.

SOLUTION

Getting Started As you go through Steps 1 through 7, be sure to sketch all asymptotes first and then plot some key points. Finally, sketch the entire graph. ▶

STEP 1: The vertical asymptote has equation $x = 3$.

STEP 2: The horizontal asymptote has equation $y = 2$.

STEP 3: $f(0) = -\frac{1}{3}$, so the y-intercept is $\left(0, -\frac{1}{3}\right)$.

STEP 4: Solve $f(x) = 0$ to find any x-intercepts.

$$\frac{2x + 1}{x - 3} = 0 \qquad f(x) = 0.$$

$$2x + 1 = 0 \qquad \text{If a fraction equals 0, its numerator must be 0.}$$

$$x = -\frac{1}{2} \qquad \text{Solve for } x.$$

The x-intercept is $\left(-\frac{1}{2}, 0\right)$.

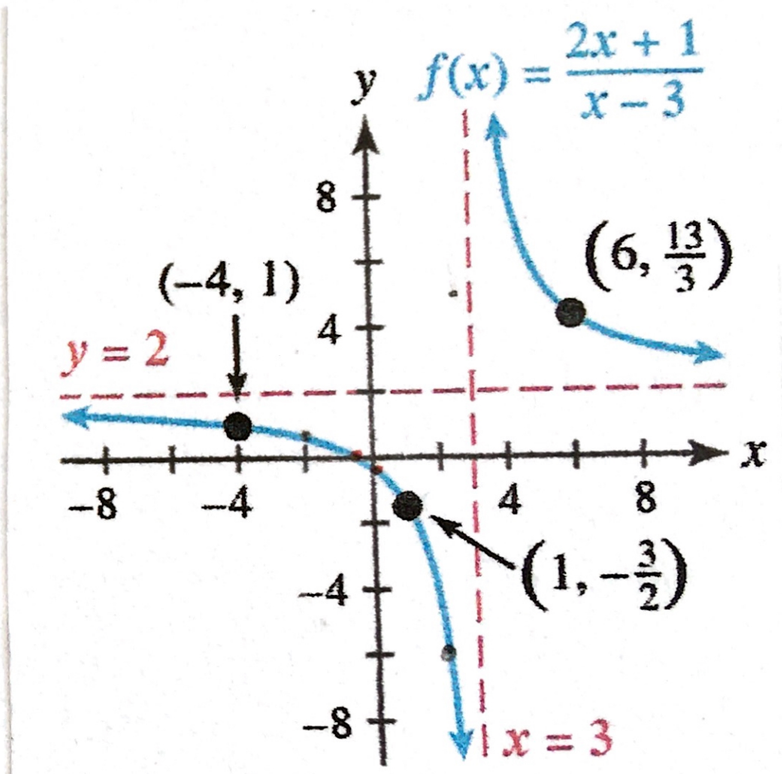


FIGURE 4.101

STEP 5: The graph does not intersect its horizontal asymptote, since $f(x) = 2$ has no solutions. (Verify this.)

STEP 6: The points $(-4, 1)$, $\left(1, -\frac{3}{2}\right)$, and $\left(6, \frac{13}{3}\right)$ are on the graph. Plot them.

x	y
2	−5
−2	3/5

STEP 7: Complete the sketch, as shown in **FIGURE 4.101**.